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Enumeration Without Repetition

HILARY PUTNAM

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ABSTRACT: An example is constructed of a recursively enumerable family of recursively enumerable sets which can not be recursively enumerated without repetitions.

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ENUMERATION WITHOUT REPETITION

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The purpose of this note is to answer the question[1]: "Can every recursively enumerable family of recursively enumerable sets[2] be enumerated without repetition[3]?" This question was, of course, stimulated by Friedberg's very surprising result that the recursively enumerable sets can be enumerated without repetition[4]. We answer the question in the negative by the following construction.

We define a family of recursively enumerable sets $A_0, A_1, A_2, \ldots$, as follows:

(i) $i \varepsilon A_i$ for all i

(ii) $2y \varepsilon A_{2y+1}$ if $y \varepsilon S$

(iii) $2y + 1 \varepsilon A_{2y}$ if $y \varepsilon S$

(iv) Nothing belongs to A_i, unless its doing so follows from (i)-(iii)

---where S is any recursively enumerable set whose complement is not recursively enumerable. Then the set A_i is recursively enumerable. (In fact, A_i has either one or two members for every i), and the relation $i \varepsilon A_j$ is also recursively enumerable, so that F is a recursively enumerable family of recursively enumerable sets. (To verify this, observe that $i \varepsilon A_j \equiv i = j \vee (Ez)(i = 2z \,\&\, j = 2z + 1 \,\&\, z \varepsilon S)$ $\vee (Ez)(j = 2z \,\&\, i = 2z + 1 \,\&\, z \varepsilon S)$. Only existential quantifiers occur in the definition (and conjunction,

disjunction, and recursively enumerable predicates); hence the predicate $i \varepsilon A_j$ is recursively enumerable). However, the family F cannot be enumerated without repetition! For suppose it could be, say as $B_0, B_1, \ldots$ where the predicate $i \varepsilon B_j$ is recursively enumerable, and $B_i \neq B_j$ for $i \neq j$. Then we could define:

(1) $y \varepsilon \overline{S} \equiv (Ex)(Ez)(2y \varepsilon B_x \,\&\, 2y+1 \varepsilon B_z \,\&\, x \neq z)$,

and hence the complement of S would be definable in terms of recursively enumerable predicates using conjunction and existential quantification alone, and so would be recursively enumberalbe, contrary to the choice of S as an recursively enumerable set whose complement is <u>not</u> recursively enumerable (i.e., a non-recursive recursively enumerable set).

To verify (1) observe that if $y \varepsilon S$, then $A_{2y} = A_{2y+1} = \{2y, 2y+1\}$ and no other A_j contains either 2y or 2y+1; so in this case the numbers 2y,2y+1 belong to one and the same of the sets $A_0, A_1, \ldots$ (note, however, that the set in question ... i.e. $\{2y, 2y+1\}$ ... is <u>repeated</u> in the enumeration $A_0, A_1, \ldots$). And if $y \varepsilon \overline{S}$, then $A_{2y} = \{2y\}$ and $A_{2y+1} = \{2y+1\}$: so in this case the numbers 2y, 2y+1 belong to <u>different</u> sets A_i. An enumeration without repetition of the sets A_i would give a proof-procedure for showing that two numbers belong to different sets A_i (since a,b belong to different sets $A_i \equiv$ a,b belong to sets B_i with different indices), and hence a way of enumerating the complement of S; it is for this reason that such an enumeration cannot exist.

Footnotes

1 This question was posed by Marion Pour-El.

2 A family of sets is called a recursively enumerable family of recursively enumerable sets if the diadic relation to $i \in A_j$ is recursively enumerable for at least one ordering $A_0, A_1, \ldots$ of the family (with or without repetition). By Kleene's "S_n^m" theorem ([2], p. 342) this is equivalent to saying that there is a recursive or even primitive recursive, function t such that $t(0), t(1), \ldots$ are gődel numbers of $A_0, A_1, \ldots$ respectively, with respect to the standard gődel numbering of all recursively enumerable sets.

3 The enumeration $A_0, A_1, \ldots$ of a family of sets F is an "enumeration without repetition" just in case (1) the diadic relation $i \in A_j$ is recursively enumerable, and (2) $i \neq j \neq A_i \neq A_j$.

4 Vide [1].

REFERENCES

[1] Richard M. Friedberg, Three theorems on recursive enumeration, Journal of Symbolic Logic, vol. 23, (1958), pp. 309-316.

[2] Stephen Cole Kleene, Introduction to metamathematics, New York, 1952.

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